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THE DYNAMICS OF A SPHERICAL PENDULUM WITH A VIBRATING SUSPENSION†

A. P. MARKEYEV

Moscow

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The motion of a spherical pendulum whose point of suspension performs high-frequency vertical harmonic oscillations of small amplitude is investigated. It is shown that two types of motion of the pendulum exist when it performs high-frequency oscillations close to conical motions, for which the pendulum makes a constant angle with the vertical and rotates around it with constant angular velocity. For the motions of the first and second types the centre of gravity of the pendulum is situated below and above the point of suspension, respectively. A bifurcation curve is obtained, which divides the plane of the parameters of the problem into two regions. In one of these only the first type of motion can exist, while in the other, in addition to the first type of motion, there are two motions of the second type. The problem of the stability of these motions of the pendulum, close to conical, is solved. It is shown that the first type of motion is stable, while of the second type of motion, only the motion with the higher position of the centre of gravity is stable. © 1999 Elsevier Science Ltd. All rights reserved.

Quite a large number of investigations have been devoted to the problem of the dynamics of a pendulum with a vibrating point of suspension (an extensive bibliography can be found, for example, in [1]). At the beginning of the present century the problem of finding whether dynamic stabilization of an inverted pendulum was possible by means of vertical vibration of its point of an inverted pendulum was possible by means of vertical vibration of its point of suspension [5] was considered in detail in [2–4]. A complete solution of the problem of the stability of vertical relative equilibria of a mathematical pendulum, the point of suspension of which executes vertical harmonic oscillations of arbitrary frequency and amplitude was obtained in [6].

1. THE HAMILTON FUNCTION

Consider a spherical pendulum, which is an absolutely solid weightless rod of length l, which performs spatial motion around one of its ends and carries a point mass m at the other end. The point of suspension O of the pendulum executes harmonic oscillations along the vertical of amplitude A and frequency Ω: $\xi_0 = A\cos\Omega t$, where ξ_0 is the displacement of the point of suspension from a certain fixed position O (Fig. 1).

We will use the spherical coordinates θ, φ as the generalized coordinates. Suppose p_θ, p_φ are the corresponding generalized momenta, which are made dimensionless by using the factor $ml^2\Omega$, while $\tau = \Omega t$ is the dimensionless time. We then have the following expression for the Hamilton function

$$H = \frac{1}{2}(p_\theta - \varepsilon^2 \sin\tau \sin\theta)^2 - \frac{1}{2}\beta^2\cos\theta + \frac{p_\varphi^2}{2\sin^2\theta} \tag{1.1}$$

$$\varepsilon = \sqrt{A/l}, \quad \beta = \sqrt{2g/(\Omega^2 l)}$$

Quantities in (1.1) that are independent of the coordinates and the momenta are dropped.

The coordinate φ is cyclical. We will write the integral corresponding to it in the form

$$2p_\varphi^2 = \alpha^2, \quad p_\varphi = \sin^2\theta d\varphi/d\tau \ (\alpha = \text{const}) \tag{1.2}$$

Ignoring the cyclical coordinate, we obtain a reduced system with one degree of freedom. Its canonical conjugate variables are the quantities θ, p_θ, while the constant α plays the role of the parameter. We will assume that $\alpha \neq 0$, since, when $\alpha = 0$, the angle φ is constant and the spherical pendulum moves in a fixed vertical plane like a mathematical pendulum.

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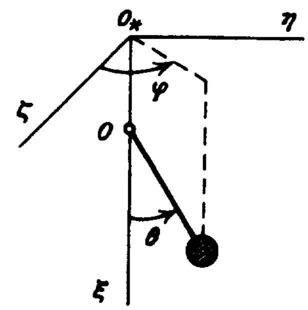

Fig. 1.

We will continue our investigation with the following assumptions. We will assume that the amplitude A of the oscillations of the point of suspension is small compared with the length l of the pendulum; this means that the quantity ε in (1.1) is a small parameter ($0 < \varepsilon \ll 1$). We will also assume that the frequency Ω of the oscillations of the point of suspension is high compared with the frequency $\sqrt{(g/l)}$ of the small oscillations of a mathematical pendulum; to fix our ideas we will assume that $\sqrt{(g/(\Omega^2 l))} < \varepsilon^2$. Moreover, we will further assume that the angular velocity of rotation of the pendulum around the vertical is small compared with Ω.

Taking these assumptions into account, we introduce the notation $\alpha^2 = \varepsilon^4 a$, $\beta^2 = \varepsilon^4 b$. The Hamilton function (1.1) can then be written in the form

$$H = \frac{1}{2}(p_\theta - \varepsilon^2 \sin\tau \sin\theta)^2 - \frac{1}{2}\varepsilon^4 b\cos\theta + \varepsilon^4 \frac{a}{4\sin^2\theta} \tag{1.3}$$

The following limits hold for the parameters a and b

$$0 < a < 2\frac{1}{g}\sin^4\theta_0\left(\frac{d\varphi}{dt}\right)_0^2, \quad 0 < b < 2$$

The subscript zero denotes the initial values of the corresponding quantities.

2. SIMPLIFICATION OF THE HAMILTON FUNCTION

To investigate the motion of the pendulum for small ε, we will use the methods of perturbation theory [7]. It is convenient first of all to make the canonical replacement of variables

$$\theta = s, \quad p_\theta = \varepsilon r \tag{2.1}$$

The following Hamiltonian corresponds to the equations of motion in the new variables

$$H = \frac{1}{2}\varepsilon r^2 - \varepsilon^2 \sin\tau r \sin s + \frac{1}{4}\varepsilon^3\left(\frac{a}{\sin^2 s} - 2b\cos s + 2\sin^2\tau \sin^2 s\right) \tag{2.2}$$

Using the canonical replacement of variables $s, r \to u, \upsilon$ we can simplify the Hamilton function (2.2) so that it does not contain the time τ in terms up to the third power of ε inclusive. To construct this replacement of variables we will use the classical perturbation theory or some modern version of it, for example, the Depri–Hori method [7]. Calculations show that the required replacement is given by the formulae

$$\begin{aligned} s &= u + \varepsilon^2 \cos\tau \sin u - 2\varepsilon^3 \sin\tau v \cos u + O(\varepsilon^4) \\ r &= v - \varepsilon^2 \cos\tau v \cos u + \frac{1}{8}\varepsilon^3(\sin 2\tau \sin 2u - 8\sin\tau v^2 \sin u) + O(\varepsilon^4) \end{aligned} \tag{2.3}$$

while the converted Hamilton function has the following form

$$H=\frac{1}{2}\varepsilon v^2+\frac{1}{4}\varepsilon^3\left(\frac{a}{\sin^2 u}-2b\cos u+\sin^2 u\right)+O(\varepsilon^4) \tag{2.4}$$

In the further calculations it is more convenient to use, instead of the variables u and υ, the canonically conjugate variables x and y, given by the equations

$$x=\cos u, \qquad v=-y\sin u \tag{2.5}$$

In the x, y variables the Hamilton function (2.4) takes the following form

$$H=\frac{1}{2}\varepsilon(1-x^2)y^2+\varepsilon^3\Pi+O(\varepsilon^4) \tag{2.6}$$

$$\Pi=\frac{1}{4}\left(\frac{a}{1-x^2}-2bx+1-x^2\right)$$

3. THE BIFURCATION CURVE

If we neglect the quantity $O(\varepsilon^4)$ in Hamiltonian (2.6), we obtain an autonomous Hamilton system, integrable in quadratures. Without dwelling on the general solution of this approximate system, we will only consider the problem of the existence of its equilibrium positions x = const, $y = 0$. The equilibrium values of x are the real roots of the equation $\partial\Pi/\partial x = 0$, which, according to the first of formulae (2.5), lie in the range $(-1, 1)$.

We will analyse the equation $\partial\Pi/\partial x = 0$ graphically. To do this, we will write it in the form $z_1(x) = z_2(x)$, where

$$z_1=\frac{x}{a}+\frac{b}{a}, \quad z_2=\frac{x}{(1-x^2)^2} \tag{3.1}$$

and in the x, z plane we will consider the graphs of the functions $z = z_1(x)$, $z = z_2(x)$.

The straight line $z = z_1(x)$ contains two positive parameters a and b, and, for values of x in the range $(-1, 1)$, intersects the curve $z = z_2(x)$ either at a single point, for which $x = x_1 > 0$, or at three points, corresponding to the three values of x_i ($i = 1, 2, 3$), which satisfy the inequalities $-1 < x_3 < x_2 < 0 < x_1 < 1$. The point P in Fig. 2 corresponds to bifurcation values of the parameters a and b. At this point the straight line $z = z_1(x)$ intersects the curve $z = z_2(x)$, i.e. at the point P the two relations $z_1 = z_2$ and $dz_1/dx = dz_2/dx$ are satisfied simultaneously. Taking (3.1) into account, these relations can be written in the form of the equations

$$f(x)\equiv x^5+bx^4-2x^3-2bx^2+(1-a)x+b=0 \tag{3.2}$$

$$g(x)\equiv 4x^3+3bx^2+b=0$$

The polynomials f and g have a common root at the critical point P. This is possible if and only if the resultant $R(f, g)$ of these polynomials is equal to zero [8]. The resultant is a function of the parameters a and b and, as calculations show, it can be represented in the form $R(f, g) = -bF(a, b)$, where

$$F(a,b)=256b^6+3(9a^2-32a-256)b^4-96(a^2-29a-8)b^2+256(a-1)^3$$

Since $b \neq 0$, the resultant vanishes only when $F(a, b) = 0$. In the a, b plane the curve $F(a, b) = 0$ is a bifurcation curve. On passing through it the number of equilibrium positions of the approximate system changes. The region $a > 0$, $2 > b > 0$ of permissible values of the parameters a and b is split by the bifurcation curve into two regions G_1 and G_3 (Fig. 3). In region G_1 there is one equilibrium position in which $x = x_1 > 0$, while in the region G_3 there are three equilibrium positions for which $x = x_i$ ($i = 1, 2, 3$), $-1 < x_3 < x_2 < 0 < x_1 < 1$.

4. PERIODIC SOLUTIONS OF THE REDUCED SYSTEM

We will now consider the reduced system with complete Hamiltonian (2.6). The approximate system

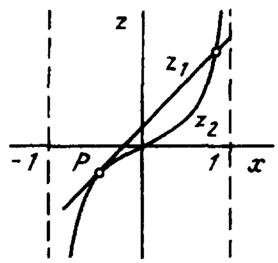

Fig. 2.

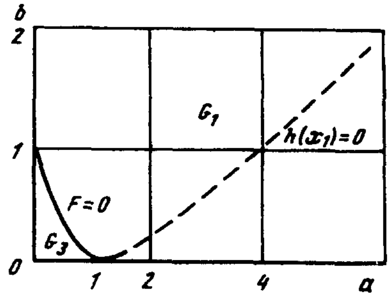

Fig. 3.

for it is unperturbed. In the neighbourhood of the equilibrium positions x = const, $y = 0$, investigated in Section 3, the complete system can be regarded as quasilinear with perturbations that are 2π-periodic in τ, the order of which with respect to ε is higher than three.

The characteristic equation of the unperturbed system, linearized in the neighbourhood of the equilibrium position, has the form

$$\lambda^2 + \varepsilon^4(1-x_i^2)\gamma_2 = 0, \quad \gamma_2 = \frac{4x_i^3 + 3bx_i^2 + b}{2x_i(1-x_i^2)} \tag{4.1}$$

Here γ_2 is the second derivative of the function Π from (2.6), calculated for $x = x_i (i = 1, 2, 3)$.

Since, for small ε, the roots of Eq. (4.1) are small, the non-resonant case of the Poincaré small-parameter method occurs. Hence [9] in the regions G_1 and G_3 (Fig. 3) from each equilibrium position, a single solution of the system with complete Hamiltonian (2.6) is produced, that is analytic in ε and 2π-periodic in τ. The periodic corrections to the equilibrium values $x = x_1, y = 0$ are of the order of ε^4 and higher.

Taking into account the replacements of variables (2.1), (2.3) and (2.5), we obtain that for the periodic motions of the pendulum considered the quantity p_θ is of the order of ε^4, while the angle θ of the pendulum with the vertical is given by the equation

$$\theta_i = \arccos x_i + \varepsilon^2\sqrt{1-x_i^2}\cos\tau + O(\varepsilon^4), \quad i = 1,2,3 \tag{4.2}$$

The quantity $O(\varepsilon^4)$ in (4.2) is 2π-periodic in τ.

5. THE MOTIONS OF A PENDULUM, CLOSE TO CONICAL, AND THEIR STABILITY

The motions of the pendulum of the initial system with two degrees of freedom, close to conical motions, correspond to periodic solutions of (4.2), of the reduced system with one degree of freedom. For these motions the angle θ_i of the pendulum with the vertical differs only slightly from its constant value $\arccos x_i$, with the corresponding dimensionless angular velocity $d\varphi_i/d\tau$ of rotation of the pendulum around the vertical is found from (1.2)

$$\frac{d\varphi_i}{d\tau} = \pm\varepsilon^2\frac{\sqrt{2a}}{2\sin^2\theta_i} = \pm\varepsilon^2\frac{\sqrt{2a}}{2(1-x_i^2)} + O(\varepsilon^4) \tag{5.1}$$

It differs from the constant value by terms of the order of ε^4, periodic in τ. The double sign in (5.1) indicates that, for the same value of x_i, two directions of rotation of the pendulum around the vertical are possible.

For the first of the solutions (4.3) we have $x_1 > 0$ and, in the corresponding motion, close to conical motion, the centre of gravity of the pendulum is below the point of suspension. This motion of the pendulum is called motion of the first type. It is the analogue of the conical motion of a spherical pendulum with a fixed point of suspension.

For the other two solutions of (4.2) we have $-1 < x_3 < x_2 < 0$. In the corresponding motions of the pendulum, close to conical motion, the centre of gravity is above the point of suspension. These motions are called motions of the second type. There is no analogue for these in the problem of the motion of a spherical pendulum with a fixed suspension.

If the values of the parameters a and b lie in the region G_3 (Fig. 3), a single motion of the first type and two motions of the second type will exist. On the bifurcation curve the two motions of the second type merge and, on passing into the region G_1, they disappear, and one motion of the first type exists in the region G_1.

We will use the stability of the motions of the pendulum, close to conical motion, with respect to perturbations of the quantities θ, p_θ. The solutions $x^* = x_i + O(\varepsilon^4), y^* = O(\varepsilon^4)$ of the equations of motion with Hamiltonian (2.6), 2π-periodic in τ, correspond to these motions. Putting $x = x^* + q, y = y^* + p$, we obtain the Hamilton function of the perturbed motion in the form of the following series in powers of q and p

$$H = H_2 + H_3 + H_4 + \ldots, \quad H_k = \sum_{\nu+\mu=k} h_{\nu\mu}(\tau) q^\nu p^\mu \tag{5.2}$$

The coefficient $h_{\nu\mu}$ are functions, 2π-periodic in τ, where, with an error of the order of ε^4, we have

$$h_{20} = \frac{1}{2}\varepsilon^3\gamma_2, \quad h_{02} = \frac{1}{2}\varepsilon(1 - x_i^2), \quad h_{30} = \frac{1}{6}\varepsilon^3\gamma_3 \tag{5.3}$$

$$h_{12} = -\varepsilon x_i, \quad h_{40} = \frac{1}{24}\varepsilon^3\gamma_4, \quad h_{22} = -\frac{1}{2}\varepsilon$$

The remaining coefficients of the forms H_k (k = 2, 3, 4) are fourth-order infinitesimals in ε. The quantity γ_2 which occurs in (5.3) is the defined by the second of Eqs (4.1), while γ_3 and γ_4 are calculated from the formulae

$$\gamma_3 = 6a\frac{x_i(1 + x_i^2)}{(1 - x_i^2)^4}, \quad \gamma_4 = 6a\frac{5x_i^4 + 10x_i^2 + 1}{(1 - x_i^2)^5}$$

We will first consider the stability in the linear approximation. We will represent the characteristic exponents of the linearized equations of the perturbed motion in the form $\lambda + O(\varepsilon^4)$, where λ is the root of Eq. (4.1). Hence, for small ε, the problem of stability in the linear approximation is determined by the sign of γ_2: when $\gamma_2 > 0$ there is stability and when $\gamma_2 < 0$ there is instability.

For motion of the first type $x_1 > 0$, and, taking into account the fact that b is positive, we obtain from (4.1) that $\gamma_2 > 0$ and, consequently, this motion is stable in the linear approximation.

Each motion of the second type is either stable in the linear approximation everywhere in its region of existence G_3 (Fig. 3), or unstable in this region. This follows from the fact that x_2 and x_3 are negative and, hence, by virtue of (4.1), the problem of stability in the linear approximation is decided by the sign of the polynomial $g(x)$ from (3.2), calculated for $x = x_i$ (i = 2, 3): when $g(x_i) < 0$ there is stability in the linear approximation and when $g(x_i) > 0$ there is instability. But the quantity $g(x_i)$ cannot vanish inside the region g_3, since otherwise the quantity $f(x_i)$ would also vanish there, and according to the results obtained in Section 3 the polynomials f and g from (3.2) only vanish simultaneously on the bifurcation curve $F(a, b) = 0$, which separates regions G_1 and G_3.

We will determine the signs of the quantities $g(x_i)$ (i = 2, 3). As discussed above, to do this it is sufficient to consider, for example, the case when a and b lie on the ray $a = b$, where $0 < a \ll 1$. We then obtain from the equation $f(x) = 0$

$$x_2 = -a + O(a^2), \quad x_3 = -1 + \sqrt{a}/2 + O(a)$$

Therefore

$$g(x_2) = a + O(a^3) > 0, \quad g(x_3) = -4 + O(\sqrt{a}) < 0$$

Hence, everywhere in the region G_3 the quantity $g(x_2)$ is positive, while the quantity $g(x_3)$ is negative.

Consequently, motion of the second type, corresponding to $x = x_3$, is stable in the linear approximation everywhere in the region G_3, while motion corresponding to $x = x_2$, is unstable.

It follows from Lyapunov's theorem on stability in the first approximation [10] that motion of the second type, corresponding to $x = x_2$, is unstable not only in the linear approximation but also in the rigorous non-linear formulation of the problem of stability.

For a rigorous solution of the problem of the stability of the motions of a pendulum of the first type and motions of the second type, corresponding to $x = x_3$, a non-linear analysis is necessary. We will carry out this analysis, basing ourselves on the KAM-theory [11, 12].

Using a Birkhoff canonical transformation $q, p \to Q, P$ we reduce the Hamilton function of the perturbed motion (5.2) to normal form [13]

$$H = \frac{1}{2}\sigma(Q^2 + P^2) + \frac{1}{4}c_2(Q^2 + P^2)^2 + O_5$$

where O_5 is a set of terms, 2π-periodic in τ, of a series, which is no less than the fifth degree in Q and P, while σ and c_2 are constants where, as calculations show,

$$\sigma = \varepsilon^2\sqrt{(1 - x_i^2)\gamma_2} + O(\varepsilon^4), \quad c_2 = -\varepsilon\frac{h(x_i)}{16\gamma_2^2(1 - x_i^2)^6} + O(\varepsilon^2)$$

$$h(x) = (3x^4 + 15x^2 - 2)a^2 + (15x^4 + 32x^2 + 1)(1 - x^2)^2 a + (2x^2 + 1)(1 - x^2)^5$$

If $c_2 \neq 0$, then, according to the Arnol'd–Moser theorem [11, 12], there is stability. Hence, for small ε, satisfaction of the inequality $h(x_i) \neq 0$ $(i = 1, 3)$ is a sufficient condition for the motions of the pendulum in question to be stable. The last condition will not be satisfied solely for those values of a and b for which $h(x_i) = 0$ and $f(x_i) = 0$ simultaneously, where f is the polynomial from (3.2).

We will consider the equality $h(x) = 0$ as a quadratic equation in a. We are only interested in positive roots of this equation. It can be shown that both roots are real and one of the roots is negative for all values of x in the range $(-1, 1)$. The second root is positive only for values of x which satisfy the inequality

$$|x| < \frac{1}{6}\sqrt{-90 + 6\sqrt{249}} \cong 0.3605 \tag{5.4}$$

We will denote this root by $a(x)$. We have

$$a(x) = \frac{15x^4 + 32x^2 + 1 + S(x)}{2(2 - 15x^2 - 3x^4)}(1 - x^2)^2 \tag{5.5}$$

$$S(x) = [3(x^2 + 3)(83x^6 + 107x^4 + x^2 + 1)]^{1/2}$$

Substituting (5.5) into the equality $f(x) = 0$ and solving it for b we obtain

$$b(x) = \frac{(x^2 + 3)(21x^2 - 1) + S(x)}{2(2 - 15x^2 - 3x^4)}x \tag{5.6}$$

Equations (5.5) and (5.6) specify in parametric form in the regions G_1 and G_3 of the a, b plane (Fig. 3) a curve on which the sufficient condition for stability $h(x_i) \neq 0$ is not satisfied.

For values of x in the range (5.4), the denominator and numerator of the fraction in expression (5.6) for $b(x)$ are positive. Hence, for a value $x = x_3 < 0$, corresponding to the second type of motion of the pendulum, the value of b is negative and, consequently, the sufficient condition $h(x_3) \neq 0$ for the stability of this motion is satisfied everywhere in the region G_3.

We will now consider the first type of motion. For this $x = x_1 > 0$. Hence, in the parametric equations of the curve $h(x_1) = 0$, on which the sufficient condition for stability is not satisfied, the parameter x is positive and satisfies condition (5.4). Taking into account the fact $b < 2$, we obtain an even narrower range of variation of the parameter: $0 < x < 0.306$. Numerical analysis shows that the curve $h(x_1) = 0$ does not pass through the region G_3. Part of this curve, lying in the region G_1, is shown in Fig. 3 by the dashed line. Outside the curve $h(x_1) = 0$ inside the regions G_1 and G_3, the first type of pendulum motion is stable. For values of a and b lying on the curve $h(x_1) = 0$ the question of the stability of this motion remains open.

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